PARTITION FUNCTIONS FOR GENERAL MULTI-LEVEL SYSTEMS

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Abstract

We describe a unified approach to calculating the partition functions of a general multi-level system with a free Hamiltonian. Particularly, we present new results for parastatistical systems of any order in the second quantized approach. Anyonic- like systems are briefly discussed.

Introduction. Statistics plays a fundamental role in determining macroscopic or thermodynamic properties of a quantum many-body system.So far there has been firm experimental evidence only for two statistics:Bose-Einstein and Fermi-Dirac.Much effort has been made to construct theories with generalized statistics and to search for deviations from Bose and Fermi statistics. Parastatistics [1-3] were the first consistent generalizations of Bose and Fermi statistics in any spacetime dimension.They are invariant under the permutation group.The N-particle states can occur in a large class of representations (REP's) of the permutation group S_N,i.e. not only in totally symmetric REP (for bosons) or totally antisymmetric REP (for fermions).Parastatistics can be related to theories obeying ordinary (Bose,Fermi) statistics and carrying suitable internal symmetries [3].Hence a natural question arises whether there are systematic ways of distinguishing between the two classes of theories.

Other generalized statistics have also been suggested,such as infinite quon statistics [4,5],a new version of parastatistics [6],fractional anyonic (braid) statistics [7,8] and other generalized statistics related to algebras of creation and annihilation operators with Fock-like representations [9-12].

To investigate physical consequences of such generalized systems,it is important to know the corresponding partition functions.There are few partial results for some parastatistical systems [13,14].Our motivation is to present a unified approach to calculating the partition function of a general multi-level system with a free Hamiltonian.Particularly,we present new results for parastatistics and a correction to results for two-level systems for parafermions given in [13]. Anyonic-like operator algebras are briefly discussed.

The partition function for a general multi-level free system. The statistical average of an observable,described by an operator $\mathcal{O}$ in a given ensemble, is defined as

$$< \mathcal{O} >= \frac{Tr\mathcal{O}\, e^{-\beta H}}{Tre^{-\beta H}} = \frac{1}{\mathcal{Z}}Tr\mathcal{O}e^{-\beta H}. \tag{1}$$

Here $\mathcal{Z}$ is the thermodynamic partition function for a multi-level system described by M independent creation (annihilation) operators $a_i^\dagger$ (a_i),$i = 1, 2, ..M$. The operator algebra is defined by a normally ordered expansion Γ (generally, no symmetries are assumed) [11]:

$$a_i a_j^\dagger = \Gamma_{ij}(a^\dagger, a), \tag{2}$$

with well-defined number operators N_i,i.e. $[N_i, a_j^\dagger] = a_i^\dagger \delta_{ij}$, $[N_i, a_j] = -a_i\delta_{ij}$ and $[N_i, N_j] = 0$ for $i, j = 1, 2, ..M$. We assume that there is a unique vacuum $|0>$ and the corresponding Fock-like representation.

The scalar product (bilinear form) is uniquely defined by $< 0|0 >= 1$, the vacuum condition $a_i|0 >= 0$, $i = 1, 2, .., M$ and Eq.(2). A general N-particle state is a linear combination of monomial state vectors $(a_{i_1}^\dagger \cdots a_{i_N}^\dagger|0 >)$, $i_1, \cdots, i_N = 1, 2, ...M$.

We consider Fock representations with no state vector of negative squared norm but we might have norm zero vectors.These null-vectors will represent the only relations between the creation (annihilation) operators on the associated (non degenerate) quotient Fock space .

For fixed sequence of indices $i_1 \cdots i_N$, we write its type as $1^{n_1}2^{n_2}...M^{n_M}$, where $n_1, n_2, ..., n_M$ are multiplicities of appearance $1, 2, ...M$ in $i_1 \cdots i_N$. Note that $n_i \geq 0$ and $\sum_{i=1}^M n_i = N$.Then, there are $N!/n_1!n_2!...n_M!$ (in principle) different states $(a_{i_1}^\dagger \cdots a_{i_N}^\dagger|0 >)$labelled by all sequences of type $1^{n_1}2^{n_2}...M^{n_M}$ and let $\mathcal{A}^\Gamma(n_1, \cdots n_M)$

be the Gram matrix of their scalar products.The number of linearly independent states among them is given by $d^{\Gamma}_{n_1,\cdots n_M} = rank[\mathcal{A}^{\Gamma}(n_1,\cdots n_M)]$, satisfying

$$0 \leq d^{\Gamma}_{n_1,\cdots,n_M} \leq \frac{N!}{n_1!n_2!...n_M!}. \tag{3}$$

All the quantities $d^{\Gamma}_{n_1,\cdots n_M}$ completely characterize the statistics,the partition function and the thermodynamic properties of the above free system realized on the corresponding Fock space.The free Hamiltonian is then

$$H_0 = \sum_{i=1}^{M} E_i \, N_i \, , \tag{4}$$

where E_i is the energy of the i^{th} level and N_i are the number operators counting particles on the i^{th} level.(Note that the statistics,i.e. the numbers $d_{n_1,\cdots n_M}$ do not determine uniquely the whole operator algebra,Eq.(2).) The partition function for a free system described by the Γ algebra, Eq.(2),is given by

$$\mathcal{Z}^{\Gamma}(x_1,...x_M) = Tr\, e^{-\beta H_0} = \sum_{N=0}^{\infty} \mathcal{Z}^{\Gamma}_N(x_1,...x_M)\, , \tag{5}$$

where

$$\mathcal{Z}^{\Gamma}_N(x_1,...x_M) = \sum_{n_1+\cdots+n_M=N} d^{\Gamma}_{n_1,\cdots n_M}\, x_1^{n_1}\cdots x_M^{n_M}. \tag{6}$$

Here $d^{\Gamma}_{n_1,\cdots n_M}$ can be considered as the degeneracy of the state with the energy $E=\sum_{i=1}^{M} n_i E_i$,where $x_i = e^{-\beta E_i}$,$\beta = \frac{1}{kT}$.

The central problem is how to compute $d^{\Gamma}_{n_1,\cdots n_M}$, $n_1,\cdots,n_M \geq 0$.The number of all allowed N-body states distributed over M energy levels is given by

$$D^{\Gamma}(M,N) = \sum_{n_1+\cdots+n_M=N} d^{\Gamma}_{n_1,\cdots n_M} \equiv \mathcal{Z}^{\Gamma}_N(\underbrace{1,1,...,1}_{M}). \tag{7}$$

The partition function for permutation invariant multi-level systems.If the relations Γ in Eq.(2) are invariant under the permutation group S_M,then the

matrix $\mathcal{A}^{\Gamma}(n_1,\cdots n_M)$ and its rank $d^{\,\Gamma}_{n_1,\cdots n_M}$ depend only on the collection of the multiplicities $\{n_1,\cdots,n_M\}$,which written in the descending order $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_M \geq 0$,$|\lambda| = \sum_{i=1}^{M} \lambda_i = N$, give rise to a partition λ of N,i.e. $d^{\,\Gamma}_{n_1,\cdots n_M} = d^{\,\Gamma}_{\lambda}$ and $\mathcal{A}^{\Gamma}(n_1,\cdots n_M) = \mathcal{A}^{\Gamma}_{\lambda}$.

If $\lambda_1 = \lambda_2 = \cdots = \lambda_N = 1$, $\lambda_{N+1} = \cdots = \lambda_M = 0$, the correspponding Young diagram,denoted by 1^N, is a column of N boxes. The corresponding $N! \times N!$ *generic* matrix is denoted by $\mathcal{A}^{\Gamma}_{1^N}$.All other matrices $\mathcal{A}^{\Gamma}_{\lambda}$, $|\lambda| = N$ for any partition λ of N are easily obtained from the generic matrix $\mathcal{A}^{\Gamma}_{1^N}$.By symmetry of Γ, it follows that $\mathcal{A}^{\Gamma}_{1^N}$ can be written as

$$\mathcal{A}^{\Gamma}_{1^N} = \sum_{\pi \in S_N} c^{\Gamma}_{\pi}\, R(\pi)\,, \tag{8}$$

where R is the right regular representation of the permutation group S_N and c^{Γ}_{π} are complex numbers. If the operators a_i and $a^{\dagger}_i$ are adjoint to each other, then the matrix $\mathcal{A}^{\Gamma}_{1^N}$ is hermitian. Of course, $d^{\Gamma}_{1^N} = rank[\mathcal{A}^{\Gamma}_{1^N}] \leq N!$. The matrix $\mathcal{A}^{\Gamma}_{1^N}$ commutes with every permutation matrix in the left regular representation. Hence the nondegenerate quotient Fock space splits into the sum of irreducible representations (IRREP's) of S_N, and we can write

$$d^{\Gamma}_{1^N} = \sum_{\mu} n^{\Gamma}(\mu)\, K_{\mu,1^N}\,, \tag{9}$$

where $n^{\Gamma}(\mu) \geq 0$ is multiplicity and $K_{\mu,1^N}$ (Kostka-Foulkes number) is the dimension of the IRREP μ of the group S_N. The sum in Eq.(9) runs over all partitions μ of N, i.e. all IRREP's. The multiplicities $n^{\Gamma}(\mu)$ can be determined from the spectrum of the matrix $\mathcal{A}^{\Gamma}_{1^N}$. ¿From Eq.(9) and S_M invariant structure of partition function

$\mathcal{Z}_N^\Gamma(x_1, ...x_M)$ it follows that

$$d_\lambda^\Gamma = \sum_\mu n^\Gamma(\mu)\, K_{\mu,\lambda}\,, \tag{10}$$

with the same $n^\Gamma(\mu)$ as in Eq.(9) and where $K_{\mu,\lambda}$ are the Kostka-Foulkes numbers enumerating semistandard(column strict) tableaux of weight λ and shape μ.Clearly, $K_{\mu,\lambda} \leq K_{\mu,1^N} = dim\mu$.The set of all $n^\Gamma(\mu)$ completely determine the statistics, the partition function and the thermodynamic properties of the free S_M invariant system. The S_M invariant partition function,Eq.(6),becomes

$$\begin{aligned}\mathcal{Z}_N^\Gamma(x_1, ...x_M) &= \sum_{\lambda;|\lambda|=N} d_\lambda^\Gamma \sum_{\pi\in S_M} x_1^{\lambda_1}\cdots x_M^{\lambda_M} = \sum_{\lambda;|\lambda|=N} d_\lambda^\Gamma\, m_\lambda(x_1, ...x_M) \\ &= \sum_{\mu;|\mu|=N} n^\Gamma(\mu) s_\mu(x_1, ...x_M),\end{aligned} \tag{11}$$

where $m_\lambda(x_1, ...x_M)$ is the monomial S_M symmetric function (summed over all distinct permutations of $(\lambda_1, ...\lambda_M)$,and $s_\mu(x_1, ...x_M)$ are the Schur's functions [15],satisfying

$$s_\mu(x_1, ...x_M) = \sum_\mu K_{\mu,\lambda}\, m_\lambda(x_1, ...x_M) \tag{12}$$

Note that for the numbers $D^\Gamma(M, N)$ of all allowable N-particle states of a permutation invariant M-level system we have:

$$\begin{aligned}D^\Gamma(M, N) &= \sum_{\lambda;|\lambda|=N} d_\lambda^\Gamma m_\lambda(\underbrace{1, 1, ...1}_{M}) = \sum_{\mu;|\mu|=N} n^\Gamma(\mu) s_\mu(\underbrace{1, 1, ...1}_{M}) \\ &= \sum_{\mu;|\mu|=N} n^\Gamma(\mu) dim\{\mu\}\end{aligned} \tag{13}$$

since $s_\mu(\underbrace{1, 1, ...1}_{M}) = dim\{\mu\}$,where $\{\mu\}$ denotes the IRREP of $SU(M)$ corresponding to the Young diagram $\{\mu\}$.

Remark. If we consider the interactions of particles described by the Hamiltonian $H = H(N_1, ..N_M)$, where N_i are number operators,then

$$\mathcal{Z}_N^{\Gamma}(H) = \sum_{n_1+\cdots+n_M=N} d^{\Gamma}_{n_1,\cdots n_M} e^{-\beta H(n_1,\ldots n_M)}. \tag{14}$$

If the algebra Γ,Eq.(2),is invariant under the permutation group S_M,then

$$\mathcal{Z}_N^{\Gamma}(H) = \sum_{\lambda;|\lambda|=N} d^{\Gamma}_{\lambda} \sum_{\pi\in S_M} e^{-\beta H(\lambda_{\pi(1)},\cdots \lambda_{\pi(M)})}. \tag{15}$$

Multi-level parastatistical systems. Here we analyze multi-level para-Bose and para-Fermi systems since they are important examples invariant under the permutation group S_M.The operator algebra corresponding to parastatistics of order p [1-3] is defined by trilinear relations

$$[a_i^{\dagger}a_j \pm a_j a_i^{\dagger}, a_k^{\dagger}] = (2/p)\delta_{jk}a_i^{\dagger}, \quad \forall i,j,k = 1,2,...,M. \tag{16}$$

Its Fock representation satisfies the following conditions:

$$< 0|0 >= 1 \qquad a_j|0 >= 0$$

$$a_i a_j^{\dagger}|0 >= \delta_{ij}|0 > \qquad i,j = 1,...M \tag{17}$$

with $|0>$ denoting the vacuum state. The upper (lower) sign coresponds to para-Bose (para-Fermi) algebra,and p is the order of parastatistics.The Fock space does not contain any state with negative squared norm if p is a positive integer [3]. The consistency condition are the following [1,3]:

$$[a_i^{\dagger}, [a_j^{\dagger}, a_k^{\dagger}]_{\pm}] = 0 \qquad \forall i,j,k, \tag{18}$$

where the upper (lower) sign coresponds to parabosons (parafermions). Note that Eq.(18) does not imply eq.(16). For $p < N$,in the N-particle space,there are additional null-states leading to relations not contained in Eq.(18).For the para-Bose and para-Fermi algebra,Eqs.(16),(17) can be presented in the form of Eq.(2), [11]. The matrices $\mathcal{A}^{p,\epsilon}_{\lambda}$, $|\lambda| = N$ can be calculated recursively using the following relation [11,12]:

$$\begin{aligned} a_i a^\dagger_{i_1} \cdots a^\dagger_{i_N}|0> &= \sum_{k=1}^{N} \delta_{ii_k} \epsilon^{k-1}\, a^\dagger_{i_1} \cdots \hat{a}^\dagger_{i_k} \cdots a^\dagger_{i_N}|0> \\ &- (2/p) \sum_{k=2}^{N} \delta_{ij_k} \sum_{l=1}^{k-1} \epsilon^l a^\dagger_{i_1} \cdots \hat{a}^\dagger_{i_l} \cdots a^\dagger_{i_{k-1}} a^\dagger_{i_l} a^\dagger_{i_{k+1}} \cdots a^\dagger_{i_N}|0> . \end{aligned} \quad (19)$$

where $\epsilon = \mp 1$, the upper (lower) sign is for parabosons (parafermions), and the $\hat{a}$ denotes omission of the corresponding operator a.

Case $M = 2$. Two-level parastatistical systems have already been studied in Ref.[13]. But the method in [13] is not completely correct (in fact the results for parafermions are false -see (20) bellow).For $M = 2$,from Eq.(16) it follows for parabosons $(a^\dagger_i)^2 a^\dagger_j = a^\dagger_j (a^\dagger_i)^2$,$i, j = 1, 2$, and for parafermions $(a^\dagger_i)^2 a^\dagger_j = 2a^\dagger_i a^\dagger_j a^\dagger_i - a^\dagger_j (a^\dagger_i)^2$,$i, j = 1, 2$.

Then every N-particle state,$N = n_1 + n_2$,$n_1 \geq n_2$, and n_1, n_2 fixed,can be expressed in terms of the following $n_2 + 1$ vectors [13]:

$$(a^\dagger_1)^{n_1}(a^\dagger_2)^{n_2}|0>; \quad (a^\dagger_1)^{n_1-1} a^\dagger_2 a^\dagger_1 (a^\dagger_2)^{n_2-1}|0>; \cdots ; (a^\dagger_1)^{n_1-n_2} \underbrace{a^\dagger_2 a^\dagger_1 \cdots a^\dagger_2 a^\dagger_1}_{2n_2}|0> .$$

In order to find the number $d^{p,\epsilon}_{\lambda}(\leq \lambda_2 + 1)$of linearly independent states of type λ ,we have studied the associate $(\lambda_2 + 1) \times (\lambda_2 + 1)$ Gram matrices up to $N = 6$.For example, for $N = 6$ there are three cases, i.e. $\lambda_2 = 1, 2, 3$ with 2×2, 3×3, 4×4 types

of matrices, respectively.We found that for parabosons (with $p \geq 2$),$d_\lambda = \lambda_2 + 1$, i.e. the result does not depend on p and coincides with the number of allowed IRREP's μ for fixed λ_1, λ_2 in the decomposition, Eq.(10). However,for parafermions $p \geq 1$ we found up to $N \leq 6$ (in contrast to [13]):

$$d_\lambda^{p,+} = \begin{cases} 0 & \text{if } p < \lambda_1 \\ p - \lambda_1 + 1 & \text{if } \lambda_1 \leq p \leq \lambda_1 + \lambda_2 = N \\ \lambda_2 + 1 & \text{if } p \geq \lambda_1 + \lambda_2 = N \end{cases} \tag{20}$$

i.e. the result does depend on p if $p < N$,in contrast to [13] and just coincides with the number of allowed IRREP's μ for fixed λ_1, λ_2 and $\lambda_1 \leq p$ in the decomposition Eq.(10). Furthermore,if number of parts in λ (the lenght of λ), $l(\lambda) \leq 2$, then $K_{\mu\lambda} = 1$ for any of the $2\lambda_2 + 1$ allowed μ, i.e. for $\mu = (\lambda_1 + \lambda_2, 0), \cdots, \mu = (\lambda_1, \lambda_2)$, [15].This means that every IRREP μ for λ, with $l(\lambda) \leq 2$,is filled only with one state.Hence, $d_\lambda^{p,\epsilon} = \sum_\mu n^{p,\epsilon}(\mu)$ and sum runs over allowed IRREP's for parabosons or parafermions.

The results for multiplicities $n^{p,\epsilon}(\mu)$ of IRREP μ in $d_\lambda^{p,\epsilon}$ Eq.(10), (checked on a computer up to $N \leq 6$, $l(\lambda) \leq 2$) are for parabosons $p \geq 1$:

$$n^{pB}(\mu) = \begin{cases} 1 & \text{if } l(\mu) \leq min(2, p) \\ 0 & \text{otherwise} \end{cases} \tag{21}$$

and for parafermions $p \geq 1$:

$$n^{pF}(\mu) = \begin{cases} 1 & \text{if } l(\mu) \leq 2 \textit{ and } l(\mu^T) \leq p \\ 0 & \text{otherwise} \end{cases} \tag{22}$$

where μ^T is the transposed tableau of μ. Note that the results, Eqs.(20-22), for a given $\lambda = (\lambda_1, \lambda_2)$ are valid for all $M \geq 2$.

Starting with Green's second quantized approach eqs.(16)-(18) and using the theorem proved in next subsection, we obtain the following :

General result:$n^{p,\epsilon}(\mu)=1$ for all allowed IRREP's μ in the decomposition of $d_\lambda^{p,\epsilon}$,eq.(10). In particular for $p \geq N = |\mu| = |\lambda|$, the multiplicities and dimensions for parabosons and parafermions coincide,i.e. $n^{pB}(\mu) = n^{pF}(\mu)$ and $d_\lambda^{pB} = d_\lambda^{pF}$.

For the partition function for two-level ($M=2$) parabosons of order $p \geq 2$ we then have:

$$\mathcal{Z}^{pB}(x_1, x_2; p) = \frac{1}{(1-x_1)(1-x_2)(1-x_1x_2)}. \tag{23}$$

i.e. it is independent of p (in agreement with [13]). The partition function for two-level ($M=2$) parafermions of order p satisfies the recurrence relation,which follows from $d_\lambda^{p,+}$, Eq.(20):

$$\mathcal{Z}^{pF}(x_1, x_2; p) = \mathcal{Z}^{pF}(x_1; p)\mathcal{Z}^{pF}(x_2; p) + x_1 \cdot x_2\, \mathcal{Z}^{pF}(x_1, x_2; p-2), \tag{24}$$

where

$$\mathcal{Z}^{pF}(x; p) = \frac{x^{p+1}-1}{x-1} \quad p \geq 0.$$

The solution of eq.(24) is given by

$$\begin{aligned}
\mathcal{Z}^{pF}(x_1, x_2; p) &= \sum_{k=0}^{[\frac{p}{2}]} (x_1x_2)^k\, \mathcal{Z}^{pF}(x_1; p-2k)\, \mathcal{Z}^{pF}(x_2; p-2k) \\
&= \frac{1}{(1-x_1)(1-x_2)} [\frac{1-(x_1x_2)^{p+2}}{1-x_1x_2} - \frac{x_1^{p+2}-x_2^{p+2}}{x_1-x_2}]. \qquad (25)
\end{aligned}$$

This result is new and differs from that in [13].In particular, we find for $p = 1, 2$:

$$\mathcal{Z}^{pF}(x_1, x_2; 1) = (1+x_1)(1+x_2),$$

$$\mathcal{Z}^{pF}(x_1, x_2; 2) = (1+x_1+x_1^2)(1+x_2+x_2^2) + x_1x_2,$$

and for $p \geq 3$, $\mathcal{Z}^{pF}(x_1, x_2; p)$ can be easily found using Eq.(24) or Eq.(25). In the limit $E_2 \to \infty$, $x_2 \to 0$,Eq.(23),(25) reduces to the partition function for one-level

parabosons (parafermions). If $p >> 1$, then $\mathcal{Z}^{pF}(x_1, x_2; p) \approx \mathcal{Z}^{pB}(x_1, x_2; p)$.However, for $p = \infty$ the para-Bose (para-Fermi) algebra (16) becomes Fermi (Bose) and hence

$$\mathcal{Z}^{pF}(x_1, x_2; p = \infty) = \frac{1}{(1-x_1)(1-x_2)} = \mathcal{Z}^{B}(x_1, x_2)$$

$$\mathcal{Z}^{pB}(x_1, x_2; p = \infty) = (1+x_1)(1+x_2) = \mathcal{Z}^{F}(x_1, x_2).$$

Generally, there is no smooth transition from generalized statistics to Bose or Fermi statistics,or from "higher" to "lower" statistics.

General case $M \geq 2$. Here we generalize the results of Ref.[13] to any multi-level system and the results of Ref.[14] to parastatistics of any order following the Green's second quantized approach, eqs.(16)-(18).

Theorem. Let us consider the N-particle states of type $1^{n_1}2^{n_2}\cdots M^{n_M}$, $\sum_{i=1}^{M} n_i = N$. If the order of Green's parastatistics defined by eqs.(16)-(18) is $p \geq N$, then the number d_λ of linearly independent physical states is

$$d_\lambda = \sum_{\mu} K_{\mu\lambda} \tag{26}$$

where $K_{\mu\lambda}$ denotes the Kostka numbers and λ is partition of N with parts $n_1, n_2, ..., n_M$.

Sketch of proof. Let us consider all possible states of the type $1^{n_1}2^{n_2} \cdots M^{n_M}$, $\sum_{i=1}^{M} n_i = N$. Our aim is to write the canonical basis for this subspace which generalize the corresponding basis for $M = 2$, Ref.[13].

We shall transform any given N-state by using the para-Bose (para-Fermi) relations, eq.(18) in the following form (as reduction rules in which, for simplicity, we abbreviate $a_k^\dagger$ to k etc.):

$$\begin{aligned} kji &= ikj + \epsilon\, jki - \epsilon\, ijk \\ kki &= -\epsilon\, ikk + (1+\epsilon)\, kik, \qquad 1 \leq i, j < k \end{aligned} \tag{27}$$

(here $\epsilon = +/-$ refers to parafermions/parabosons). We first start shifting the rightmost index M to the right. This will end with M either in the last or next to last right position $*M$ or $*Mi$, $i < M$. Then we apply the same procedure to the second rightmost index M by shifting it to the right as before. After moving all of M's, we end up with states like $*M^{n_M}$, $*M\ i_1\ M^{n_M-1}, \cdots, *M\ i_1\ M\ i_2 \cdots M\ i_{n_M}$, i.e. with $n_M + 1$ types of states.

By applying the approach of Bergman, Ref.[16] to the relations (27), we find new important relations (no further relations except these exist if $p \geq N$)

$$kjki = kikj + jkik - ikjk, \quad k > j \geq i, \tag{28}$$

which together with (27) ensure the diamond condition for the para-Fermi/para-Bose algebras. By applying (28) to the above types of states we further reduce them to the states satisfying $1 \leq i_1 \leq i_2 \leq \cdots \leq i_{n_M} < M$.

After exhausting M's we perform the same procedure on $(M-1)$'s , $(M-2)$'s ,.. and finally on 2's. In this way we obtain a canonical basis which generalizes the basis for two-level system of the previous subsection. We point out that the dimensions d_λ satisfy the same recursion relation as coefficients a_λ, ($a_\lambda = \sum_\mu K_{\mu\lambda}$) in the expresion $\prod_{i=1}^M 1/(1-x_i)\ \prod_{i<j}^M 1/(1-x_i x_j) = \sum_\lambda a_\lambda m_\lambda$. Hence, we conclude that the eq.(26) in the theorem is proved. (Alternatively, one obtains one to one correspondence between the states in canonical basis and terms in the expansion of $\sum_\mu s_\mu(x_1, ..., x_M)$.)

Comparing the eq.(26) with eq.(10), we obtain $n(\mu) = 1$ for all allowed IRREP's μ, i.e. $l(\mu) \leq M$ for para-Bose and para-Fermi case.

Furthermore, if $p < N$ then the Green's parastatistics imply that only IRREP's

with $l(\mu) \leq p$ for para-Bose and $l(\mu^T) \leq p$ for para-Fermi are allowed. These are additional restrictions on the multiplicities $n(\mu)$ in eq.(10). We point out that these restrictions are not put by hand, but they are intrinsicaly contained in eq.(16). The complete proof of the theorem and corrolaries will be presented elsewhere.

This theorem enables a simple generalization of Eq.(23).The partition function for multi-level parabosons for $p \geq M$ is:

$$\mathcal{Z}^{pB}(x_1, ...x_M; p \geq M) = \sum_{\lambda} s_{\lambda}(x_1, ...x_M) = \prod_{i=1}^{M} \frac{1}{(1-x_i)} \prod_{i<j}^{M} \frac{1}{(1-x_i x_j)}. \tag{29}$$

The identity used in Eq.(29) can be found in [15].The results for parastatistical systems of order $p = 2$ [14] can be easily obtained from Eqs.(10,11) and our eq.(26). We point out that the parastatistics in the first quantized approach of Ref.[17], [18] leads to the same partition functions as we have found. However, these two parastatistics basicaly differ: (i) since within the first quantized approach [17],[18], the Green's trilinear commutation relations , eq.(18) cannot be obtained, and (ii) the probabilites of finding IRREP Γ_c in $\Gamma_a \otimes \Gamma_b = \sum_c \Gamma_c$ and finding $\Gamma_a \otimes \Gamma_b$ in $\Gamma_c = \sum_{(a,b)} \Gamma_a \otimes \Gamma_b$ calculated within the approach of [17],[18] would be completely different from those calculated in the second quantized approach. Hence these two approaches are not equivalent. The complete analysis will be published separately.

$\mathbf{Anyonic-like\,multi-level\,systems}$. Here we study another class of multi-level systems described by algebras with Fock representations,which possess anyonic-like statistics and are not invariant under the permutation group. We do not assume any symmetry requirements . Instead, we assume for $i \neq j$:

$$a_i a_j^{\dagger} = e^{i\phi_{ij}} a_j^{\dagger} a_i, \qquad \phi_{ij} = -\phi_{ji} \in \mathbf{R}, \qquad i \neq j, \quad i, j = 1, 2...M \tag{30}$$

and a_i, $a_i^{\dagger}$ are adjoint to each other.It is easy to show that $d_{n_1, \cdots n_M} = 1$ for any set

of mutually different indices $i_1, \cdots i_N$. In this case, any monomial state consisting of permuted indices $i_1, \cdots i_N$ is ,up to unit phase, equal to $(a^\dagger_{i_1} \cdots a^\dagger_{i_N}|0>)$. Therefore, we find the anyonic-like commutation relations [5]:

$$a^\dagger_i a^\dagger_j = e^{-i\phi_{ij}} a^\dagger_j a^\dagger_i, \qquad a_i a_j = e^{-i\phi_{ij}} a_j a_i \qquad i \neq j. \tag{31}$$

We have assumed no specific relation $a_i a^\dagger_i = \Gamma(a^\dagger, a)$ for the same indices. The independent number operators N_i are assumed. Then for a multi-level anyonic system with the free Hamiltonian $H_0 = \sum_{i=1}^M E_i\, N_i$, the partition function is

$$\mathcal{Z}(x_1, ...x_M; \phi_{ij}) = \prod_{i=1}^M \mathcal{Z}(x_i, p_i), \tag{32}$$

where $\mathcal{Z}(x;p) = (1 - x^{p+1})/(1 - x)$ is the one-level partition function of a single (parafermion-like) oscillator satisfying $a^{p+1} = 0$, $p \in \mathbf{N}$. Such a system is physically equivalent to the set of M commuting (or anticommuting) single-mode generalized oscillators $\tilde{a}^\dagger_i$, $\tilde{a}_i$. The two sets of operators $\{a_i\}$ and $\{\tilde{a}_i\}$ are connected by a generalized Jordan-Wigner transformation [19].

Starting from Haldane's definition of generalized exclusion statistics [20], Karabali and Nair [21], using additional assumptions derived an operator algebra of the type given by eq.(30) with $\phi_{ij} = \frac{\pi}{p+1} sign(i - j)$, $i \neq j$ and $a_i^{p+1} = 0$, for $\forall i$. From our results it follows that the partition function corresponding to the system described by the Karabali-Nair algebra with the free Hamiltonian $H_0 = \sum_{i=1}^M E_i\, N_i$ is

$$\mathcal{Z}(x_1, ...x_M; p) = \prod_{i=1}^M \frac{1 - x_i^{p+1}}{1 - x_i}. \tag{33}$$

The system with this partition function was analyzed in [22]. However it does not reproduce the Haldane-Wu interpolating distribution.

Finally,let us mention algebras with no independent number operators N_i, but with a well-defined total number operator N , $[N, a_i^\dagger] = a_i^\dagger$, $[N, a_i] = -a_i$ for $i, j = 1, 2, ..M$. Let us consider a free Hamiltonian of the form $H_0 = E\,N$. Then the partition function is

$$\mathcal{Z}^\Gamma(x_1, ...x_M) = \sum_{N=0}^{\infty} D^\Gamma(M, N) e^{-\beta EN}, \tag{34}$$

where $D^\Gamma(M, N)$ is number of independent states $(a_{i_1}^\dagger \cdots a_{i_N}^\dagger |0>)$, $i_1, \cdots, i_N = 1, ..., M$. If $a_i a_j = a_j a_i$, then $0 \leq D(M, N) \leq D^B(M, N)$. Such an example is the system derived from the Calogero-Sutherland model analyzed in [23]. A complete analysis including thermodynamic properties of this and other generalized multi-level systems will be treated separately.

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